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THE CONSTRUCTION OF SPECIAL FUNCTIONS FOR THE ANALYTIC SOLUTION OF BOUNDARY-VALUE PROBLEMS OF THE THEORY OF CONICAL SHELLS†

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A linearly independent fundamental system of solutions of the ordinary differential equations of the deformation of layered orthotropic conical shells is constructed for computer calculations. The fundamental system of solutions obtained enables boundary-value problems for shells with arbitrary parameters to be solved analytically.

Many fundamental systems of solutions for thin-walled structures obtained analytically and analytic algorithms for solving boundary-value problems have ceased to be used since they turn out to be practically linearly dependent for computer calculations, and the calculation becomes unstable. The reason for the instability of the calculations in this case is the fact that there are rapidly varying functions among the integrals of the differential equations considered. If there are one or several rapidly varying functions in each solution or any part of the fundamental system of solutions they will predominate for large values of the argument, and such a fundamental system of solutions becomes practically linearly dependent, while the matrix of the system of linear algebraic equations for determining the arbitrary constants from the boundary conditions is ill conditioned.

Since there are different fundamental systems of solutions for one system of ordinary differential equations, the problem arises of constructing that fundamental system of solutions which remains linearly independent during calculations for arbitrary parameters of thin-walled elements. This problem is solved below for a conical shell.

For isotropic conical shells with non-axisymmetric loading, solutions were constructed in [1] in terms of special functions for ordinary differential equations, obtained after separation of the variables of Fourier's methods. For the zeroth harmonic of the expansion, i.e. for axisymmetric deformation, and for the first harmonic, that is, for antisymmetric deformation, for shells of constant thickness the solutions were expressed in [1] in terms of well-known Thomson functions of the first and second kind, which are convenient to use by virtue of two facts. First, for any parameters of the conical shell, to calculate the Thomson functions two methods in all are sufficient: for small values of the independent variable one can use power series or an expansion in orthogonal polynomials, while for large values of the variable one can use asymptotic expansions in the neighbourhood of a point at infinity. As a rule, the region in which these methods can be used is limited, but in this case, by combining them, one can obtain sufficient accuracy for any values of the independent variable. Second, the fundamental system of solutions of ordinary differential equations constructed using Thomson functions is linearly independent in the neighbourhood of a point at infinity and can be functionally normalized, as proposed in [2]. Hence, to calculate the stress–strain state of an axisymmetrically and antisymmetrically deformed isotropic conical shell of constant thickness one only needs to normalize the solution [1], and it can then be used for practically any geometrical parameters of the shell.

The solution of the problem for other harmonics, i.e. for cyclic deformation of a conical isotropic shell of constant thickness, was expressed in [1] in terms of generalized hyperbolic functions. The fundamental system of solutions constructed in this way is only suitable for calculations in the neighbourhood of the vertex of the shell, i.e. for small values for the independent variable. For large values of the independent variable, which arise in practical problems, one cannot use this system of solutions because of the above-mentioned predominance of rapidly increasing solutions.

We will consider the problem of choosing the fundamental system of solutions of the solving equation of the problem of cyclic deformation of a conical shell and we will apply the results obtained to the more general case of a laminated orthotropic shell.

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After separating the variables in the equations of statics, we have a system of ordinary different equations [3]. Following [1], we apply a complex transformation to them and obtain a single fourth-order solving equation

$$\left\{\delta^4+2\delta^3+\delta^2[1-\xi^2(1+D_\Delta)]-2\xi^2D_\Delta\delta+D_\Delta\frac{k^2-1}{k^2}\xi^4-z(\delta^2+3\delta+2)\right\}N(z)=0 \quad (1)$$

$$\delta=z\frac{d}{dz},\quad D_\Delta=\frac{D_{22}}{D_{11}},\quad \xi=\frac{k}{\sin\gamma}$$
$$z=is\operatorname{ctg}\gamma\sqrt{\frac{C_{11}C_{22}-C_{12}^2}{C_{11}D_{11}}} \quad (2)$$

where $N(z)$ is the required function, z is the new variable, C_{ij} and D_{ij} are generalized stiffnesses of the laminated material, s is the length of the meridian measured from the vertex of the middle surface, γ is the angle between the meridian and the axis of the middle surface and k is the number of the term of the Fourier series in the expansion in a circular coordinate.

We will represent Eq. (1) in the form [4]

$$\left[(-1)^\tau z\prod_{j=1}^{p}(\delta-a_j+1)-\prod_{j=1}^{q}(\delta-b_j)\right]N(z)=0,$$
$$\tau=0,\quad p=2,\quad q=4,\quad a_1=0,\quad a_2=-1 \quad (3)$$

The parameters b_j ($j = 1, \ldots, 4$) are the roots of the governing equation for (1). It was shown in [1] that the parameters b_j are real quantities for an isotropic shell. In the case of a laminated orthotropic shell they will usually be complex quantities.

Equation (3) has a regular singular point $z = 0$ and an irregular singular point $z = \infty$. Its integrals can be expressed in terms of the generalized hyper-geometric function ${}_pF_q(\alpha_p;\ \rho_q \mid z)$ and the Meyer function $G_{p,q}^{m,n}(z \mid a_p;\ b_q)$. These functions are written in abbreviated form: for these parameters we will write, for example, b_q instead of $b_1, b_2, \ldots, b_q$ and similarly $\Pi_{j=1}^{p}\Gamma(a_j + k)$ will be denoted by $\Gamma(a_p + k)$, etc.

The solution derived in [1] uses the fundamental system of solutions in the neighbourhood of the point $z = 0$. In the case considered here the analogous system of functions is

$$N_h(z)=\exp(i\pi b_h)G_{2,4}^{1,2}\left(z\exp(-i\pi)\left|\begin{matrix}a_p\\ b_h,b_1,\ldots,b_{h-1},b_{h+1},\ldots,b_q\end{matrix}\right.\right)=$$
$$=\frac{\Gamma(1+b_h-a_p)}{\Gamma(1+b_h-b_q)}z^{b_h}{}_2F_3\left(\begin{matrix}1+b_h-a_p\\ 1+b_h-b_q^*\end{matrix}\middle| z\right),\quad h=1,\ldots,4 \quad (4)$$

Here and henceforth the asterisk denotes that the term $1 + b_h - b_q$ has been omitted if $h = q$.

In (2) we have under the radical sign the ratio of the flexural stiffness of the shell to the membrane stiffness, i.e. a quantity proportional to the square of the shell thickness h. Hence, the modulus of the variable z is proportional to the ratio s/h and can be extremely large. For such values of the variable, when analysing the behaviour of the special functions considered, it is best to use their asymptotic expansions in the neighbourhood of a point at infinity.

For the function ${}_2F_3(\alpha_p;\ \rho_q \mid z)$ the expansion as $|z| \to \infty$ has the form [4]

$${}_2F_3\left(\begin{matrix}\alpha_p\\ \rho_q\end{matrix}\middle| z\right)\sim\frac{\Gamma(\rho_q)}{\Gamma(\alpha_p)}\{K_{2,3}(z)+K_{2,3}(ze^{i2\pi})+L_{2,3}(ze^{i\pi})\} \quad (5)$$

$-3\pi + \delta \leqslant \arg z \leqslant \pi - \delta,\ \delta > 0.$

As $|z| \to \infty$ the function $K_{2,3}(z)$ tends exponentially to infinity, $K_{2,3}(ze^{i2\pi})$ tends exponentially to zero, and the change in $L_{2,3}(ze^{i2\pi})$ has an algebraic form. Consequently, $K_{2,3}(z)$ will be the predominant function, i.e. for large values of the variable $K_{2,3}(ze^{i2\pi})$ and $L_{2,3}(ze^{i2\pi})$ will be negligibly small compared with $K_{2,3}(z)$. Further, it follows from the procedure [4] for determining the coefficients of asymptotic series $K_{p,q}(z)$, that for all solutions from (4) the expressions $z^{b_h}K_{p,q}(z)$ are one and the same quantity. We finally obtain that, for large values of the variable, the solutions (4) only differ from one another in practice by a constant factor and this fundamental system of solutions is linearly dependent with a very high degree of accuracy.

The fundamental system of solutions constructed in the neighbourhood of the irregular singular point $z = \infty$ does not have this drawback. Following [4], we will set up this system of solutions.

We introduce the following notation

$$\sigma = q - p = 2; \quad \varepsilon = 1; \quad \nu = q - p - \tau = 2 \tag{6}$$

We choose integers λ and ω such that

$$|\arg z + (\nu - 2\lambda + 1)\pi| < (\sigma/2 + 1)\pi$$

$$|\arg z + (\nu - 2\psi)\pi| < (\sigma + \varepsilon)\pi, \quad \psi = \omega, \omega + 1, \ldots, \omega + \sigma - 1$$

These conditions are satisfied by $\lambda = 1$, $\omega = 0$ and by two values of ψ, namely, zero and one. Then the fundamental system of solutions of Eq. (3) can be compiled from the following functions.

1. The two functions

$$G^{q,0}_{p,q}\left(z\exp[i\pi(\nu - 2\psi)] \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right)$$

corresponding to $\psi = 0$ and $\psi = 1$, i.e.

$$G^{4,0}_{2,4}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right) \text{ and } G^{4,0}_{2,4}\left(ze^{2i\pi} \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right) \tag{7}$$

The asymptotic series [4] for these solutions, apart from a factor, are $z^{b_h}K_{p,q}(z)$ and $z^{b_h}K_{2,3}(ze^{i2\pi})$, respectively ($b_h$ is any of the parameters $b_i, \ldots, b_2$) i.e. one of these functions tends to zero as $|z| \to \infty$ and the other tends to infinity.

2. Since the parameters a_1 and a_2 differ from one another by an integer, only one of the two functions

$$G^{q,1}_{p,q}\left(z\exp[i\pi(\nu - 2\lambda + 1)] \,\middle|\, \begin{matrix} a_t, a_1, \ldots, a_{t-1}, a_{t+1}, \ldots, a_p \\ b_q \end{matrix} \right), \quad t = 1, \ldots, p \tag{8}$$

can occur in the fundamental system of solutions, for example, corresponding to $t = 2$

$$G^{4,1}_{2,4}\left(ze^{i\pi} \,\middle|\, \begin{matrix} a_2; a_1 \\ b_q \end{matrix} \right) \tag{9}$$

Its asymptotic expansion in the neighbourhood of a point at infinity has the form

$$G^{4,1}_{2,4}\left(ze^{i\pi} \,\middle|\, \begin{matrix} a_2; a_1 \\ b_q \end{matrix} \right) \sim \frac{z^{a_2 - 1}\Gamma(1 + b_q - a_2)}{\Gamma(1 + a_p - a_2)}\, {}_4F_1\left(\begin{matrix} 1 + b_q - a_2 \\ 1 + a_p - a_2^* \end{matrix} \,\middle|\, \frac{1}{z} \right) \tag{10}$$

There is a fairly lengthy procedure, described in [4], for determining the other function. As a result we obtain a solution which has a logarithmic singularity at infinity. However, it is more convenient to obtain the necessary solution by considering the asymptotic expansion (5). The function $L_{r,s}(z)$ in this formula is defined as follows:

$$L_{r,s}(z) \sim \sum_{t=1}^{r} \frac{z^{\alpha_t - 1}\Gamma(\alpha_t)\Gamma(\alpha_r - \alpha_t)^*}{\Gamma(\rho_s - \alpha_t)} \, {}_{s+1}F_{r-1}\left(\begin{matrix} \alpha_t, \; 1+\alpha_t - \rho_s \\ 1+\alpha_t - \alpha_r^* \end{matrix} \middle| \frac{(-1)^{s-r}}{z} \right) \tag{11}$$

If we use expansion (5) to calculate the functions (4), it can be seen that the asymptotic series for the three constructed solutions (7) and (9) occur in the expression obtained with certain coefficients. In (5) there is also a fourth term corresponding to $t = 1$ in (11). It must be calculated by taking the limit [4], since a_1 and a_2 differ from one another by an integer. As a result a logarithmic solution is obtained.

From these considerations we choose as the fourth solution the required fundamental system of solutions

$$N_1(z) - N_2(z) \tag{12}$$

or the difference between any of the other functions from (4). As can be seen from (4) and (5), the asymptotic expansion for (12) contains no predominant function $K_{2,3}(z)$ and is practically a linear combination of (10) and (11), since the quantity $K_{2,3}(ze^{i2\pi})$ is negligibly small compared with $L_{2,3}(z^{i\pi})$. Since slowly varying functions occur in (10) and (11), no essential difficulties arise in using (12) in the calculations.

Thus, (7), (9) and (12) are convenient for calculating the fundamental system of solutions of Eq. (3). It can be shown that this system of functions is linearly independent. Its functional normalization is similar to that proposed in [2]: one must take as the normalizing factor for functions of increasing modulus their values on the right-hand end of the shell, and their values on the left-hand end for functions of decreasing modulus. Then, when determining the arbitrary constants the matrix of the system of linear algebraic equations will be will conditioned for as long a length of the conical shell as desired.

Naturally, the solution obtained turns out to be more effective than sweep methods, since they lead neither to an increase in the order of the system considered nor to splitting of the shell into individual parts, nor to the use of additional computing procedures involving continuous or discrete orthogonalization of the solutions [5, 6].

As an example for the practical application of this method consider an isotropic conical shell closed in an end of the larger radius ($R_2/h = 1500$). At the free end ($R_1/h = 1000$) a load is applied which is distributed over an arc of 2° in the direction towards the shell axis. The length of the shell $x_{max}/h = 1000$ (x is the distance along the shell axis measured from the left end and h is the shell thickness). Figure 1 shows a graph of the change in the moments M_1 and M_2 along the generatrix in the region of application of the load. The results are represented in dimensionless form (in fractions of the product of the load intensity and the shell thickness). This example demonstrates the stability of the calculations and the fact that the use of an analytic method of solving boundary-value problems is promising.

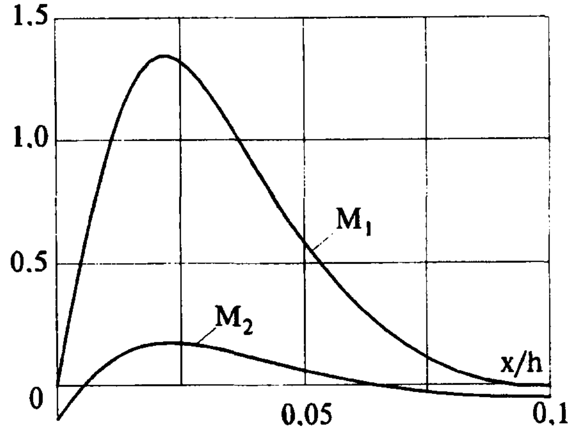

Fig. 1.

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